

## Study of the surface conductance coefficient in a plastic bar

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### Abstract

The surface conductance coefficient for the polymethyl methacrylate plastic was studied with accurate experimental measurements on a bar of this material. It was found that the coefficients for the steady state and for cooling are different, the second being about 22% greater than the first. This result contrasts with that from an analogous study made for an iron bar under the same experimental conditions, where no difference was found between the steady state and cooling.

### 1. INTRODUCTION

In the same way that Fourier's law of thermal conduction expresses a linear relationship between heat flow and the driving force in the form of a temperature gradient, the proportionality factor being the thermal conductivity  $K$ , Newton's law of cooling stated a linear relationship between the heat flow from a medium to the environment maintained at a lower constant temperature, the proportionality factor  $H$  being known as the surface conductance or surface heat transfer coefficient. However,  $H$  was soon found not to be a constant, since it depends on many parameters defining the heat transfer process. The behaviour of  $H$  will depend on the particular case studied: the shape and nature of the material and the temperature difference between the material and the environment each play an important role in the calculations of surface conductance. The  $K$  and  $H$  coefficients are strongly related for any given temperature distribution in a medium. The ratio  $H/K$  is proportional to the Biot number, and determines whether the temperature gradient in the steady state is sharper or smoother, or the temperature decay during cooling is faster or slower.

There are many well-known experimental methods to determine the thermal conductivity in different media: solids, liquids, poor conductors, metals, etc. They may be classified essentially as steady state, periodic heating, and variable state methods [1]. In contrast, there are no accurate rules for determining  $H$ , and approximate values must be used. In practice

a few numerical values are taken as given, enabling the order of magnitude of  $H$  to be estimated roughly in a practical problem.

In a previous paper [2] an experimental method was given for determining  $H$  in a metal bar from the temperature behaviour in the steady state and during cooling [3]. The present study extends and adapts this method to the analysis of data for a bar of the plastic polymethyl methacrylate (PMMA), for which the temperature distribution [4] in both the steady state and cooling regimes was found to be different from that for the metal bar. In Section 2 the mathematical basis of the method is described with the solutions of the heat transfer equation. The experimental results for the steady state and for cooling are studied in Section 3, and a discussion and conclusions are presented in Section 4.

## 2. MATHEMATICAL SOLUTION

The one-dimensional differential equation for the heat flow in an isotropic homogeneous bar, with no internal source of heat and with constant geometrical, physical and thermal properties, is

$$\frac{\partial T}{\partial t} = \frac{K}{\rho c} \frac{\partial^2 T}{\partial x^2} - \frac{Hp}{c\rho w} (T - T_0) \quad (1)$$

where  $p$ ,  $\rho$ ,  $w$  and  $c$  are the perimeter, density, cross section and specific heat of the bar. The temperature of the environment is  $T_0$ , usually taken as zero, so that  $T$  should be understood as the temperature excess of the material over the environment.

Particular cases of eqn. (1) are the steady state,  $(\partial T/\partial t) = 0$ , and cooling,  $(\partial^2 T/\partial x^2) = 0$ , with the respective equations

$$\frac{d^2 T}{dx^2} - \frac{Hp}{Kw} T = 0 \quad (2)$$

and

$$\frac{dT}{dt} - \frac{Hp}{c\rho w} T = 0 \quad (3)$$

The analytical solutions of eqns. (2) and (3) are the single exponentials

$$T_i = T_1 \exp\left(-\sqrt{\frac{Hp}{Kw}} x\right) \quad (4)$$

$$T' = T_i \exp\left(-\frac{Hp}{c\rho w} t\right) \quad (5)$$

where  $T_1$  is the temperature of the first point on the bar and  $T'$  denotes the cooling of a point  $i$  of the bar which started from the steady state at temperature  $T_i$ .

The exponents in eqns. (4) and (5) are usually written as

$$m^2 = \frac{Hp}{Kw} \quad (6)$$

$$m' = \frac{Hp}{c\rho w} \quad (7)$$

As  $m$  and  $m'$  are experimental values obtained from the steady state and cooling, if  $K$  is known, eqns. (6) and (7) give two ways of calculating  $H$ . With the reasonable supposition, verified experimentally for the iron bar [2], that the surface conductance is the same in the steady state and in cooling, eqns. (6) and (7) give the general relationship

$$m' = \frac{K}{\rho c} m^2 \quad (8)$$

Thus, for any given material, it is possible to know not only  $H$ , but also the conductivity, the thermal diffusivity  $\alpha = K/\rho c$ , and the Biot number  $Bi = HL/K$ , where  $L = r_0/2$  for a cylinder of radius  $r_0$  [5].

### 3. RESULTS

The experimental procedure has been fully described in ref. 4, and only the study of the new experimental results will be treated here.

The parameter  $H$  is very difficult to measure because it depends on many variables, each with its own source of error. One must therefore take the best solution to the differential equation for the temperature distribution in the bar, i.e., the solution whose constants have been calculated with the smallest statistical error. For PMMA [4,6], double exponential solutions were found to be the best governing equations for the steady state and for cooling. Thus, the true (experimental) solution of eqns. (2) and (3) are not eqns. (4) and (5), but

$$T_i = T_{1i} \exp(-m_1 x) + T_{2i} \exp(-m_2 x) \quad (9)$$

and

$$T' = T'_{1i} \exp(-m'_1 t) + T'_{2i} \exp(-m'_2 t) \quad (10)$$

with the auxiliary conditions

$$T_{1i} + T_{2i} = T'_{1i} + T'_{2i} = T_i \quad (11)$$

$$m = m_1 - m_2; (m_1 > m_2) \quad (12)$$

and

$$m' = m'_1 - m'_2; (m'_1 > m'_2) \quad (13)$$

TABLE 1

Steady state cases studied (I–XI) in the PMMA bar, where L and C mean heated by a light or by an electric coil respectively. The temperature excess at the first point of the bar  $T_1$  is taken as reference. The parameters in the fit are those of eqn. (9). The last column is the sum of the squares of the differences between the experimental and expected values

| Case | Heat | $T_1$ | $(T_{11} \pm \sigma)$ | $(-m_1 \pm \sigma)$ | $(T_{21} \pm \sigma)$ | $(-m_2 \pm \sigma)$ | $\sigma^2 x^2$ |
|------|------|-------|-----------------------|---------------------|-----------------------|---------------------|----------------|
| I    | L    | 71.0  | 62.9 ± 0.8            | 41.26 ± 0.08        | 7.79 ± 0.08           | 4.54 ± 0.07         | 3.02           |
| II   | C    | 52.0  | 51.6 ± 0.8            | 40.39 ± 0.08        | 0.51 ± 0.08           | 1.99 ± 0.07         | 1.07           |
| III  | L    | 40.8  | 37.6 ± 0.7            | 32.70 ± 0.07        | 3.36 ± 0.08           | 1.44 ± 0.06         | 1.31           |
| IV   | C    | 33.3  | 30.7 ± 0.7            | 32.63 ± 0.07        | 2.60 ± 0.07           | 0.74 ± 0.06         | 1.06           |
| V    | L    | 25.9  | 23.8 ± 0.7            | 32.68 ± 0.07        | 2.31 ± 0.07           | 1.11 ± 0.06         | 0.91           |
| VI   | L    | 24.0  | 20.9 ± 0.9            | 48.47 ± 0.08        | 3.11 ± 0.08           | 4.46 ± 0.07         | 0.11           |
| VII  | C    | 22.7  | 21.7 ± 0.8            | 39.18 ± 0.08        | 1.08 ± 0.08           | 2.07 ± 0.07         | 0.42           |
| VIII | C    | 22.0  | 20.5 ± 0.7            | 32.36 ± 0.07        | 1.53 ± 0.07           | 1.05 ± 0.06         | 0.44           |
| IX   | L    | 22.0  | 20.4 ± 0.8            | 35.91 ± 0.08        | 1.69 ± 0.08           | 0.78 ± 0.06         | 0.43           |
| X    | L    | 21.7  | 19.1 ± 0.8            | 32.95 ± 0.07        | 2.63 ± 0.07           | 0.50 ± 0.06         | 0.65           |
| XI   | C    | 19.0  | 17.5 ± 0.8            | 37.70 ± 0.08        | 1.61 ± 0.08           | 0.85 ± 0.07         | 0.45           |

Table 1 shows the results of fitting eqn. (9) to the experimental data for the eleven cases studied in the steady state. The first point in the bar ( $i = 1$ ) has been taken as reference because it covers the greatest range of temperature, cooling down from 71°C to 19°C. The standard deviations  $\sigma$  of the temperature ( $T_{11}$  and  $T_{21}$ ) and exponent ( $m_1$  and  $m_2$ ) parameters are given next to the values themselves. The results of the fit are quite good, as can be seen from the last column, where  $\sigma^2 x^2$  denotes the sum of the squares of the differences between the experimental and predicted data [7]. The values calculated for the temperature parameters verify exactly, for all the cases, the first part of eqn. (11) for  $i = 1$ , that is

$$T_{11} + T_{21} = T_1 \quad (14)$$

in spite of the large differences between these parameters: for most cases  $T_{11}$  is 6.7–11.8 times greater than  $T_{21}$ , even up to 20 (case VII) and 101 (case II) times greater. The values for the exponent parameters show that  $m_1$  is always much larger than  $m_2$ , but there is no particular dependence on temperature or manner of heating. The smallest and largest differences between them are about 9 and 66 times for the cases I and X respectively. This means that the first exponential in eqn (9) governs the main behaviour of the temperature distribution in the steady state, the other being a correction or fine adjustment with a contribution lying between 1.5% (case X) and 11% (case I).

Table 2 shows the results of fitting eqn. (10) to the experimental cooling data. As can be seen, not all the cases studied in the steady state have been taken into account in the cooling regime for reasons of accuracy and brevity. The cooling of the first steady state has been omitted because it

TABLE 2

Cooling chosen from among the steady state points studied in Table 1. The time of cooling  $t$  is expressed in minutes and  $N$  is the number of measurements taken during the cooling. The parameter fitting is with eqn. (10)

| Case | $t(\text{min})$ | $N$ | $(T'_{11} \pm \sigma)$ | $(-m'_1 \pm \sigma) \times 10^{-3}$ | $(T'_{21} \pm \sigma)$ | $(-m'_2 \pm \sigma) \times 10^{-3}$ | $\sigma^2 x^2$ |
|------|-----------------|-----|------------------------|-------------------------------------|------------------------|-------------------------------------|----------------|
| II   | 188             | 38  | $31.6 \pm 1.2$         | $0.81 \pm 0.06$                     | $21.83 \pm 0.03$       | $0.39 \pm 0.02$                     | 4.08           |
| III  | 86              | 57  | $29.1 \pm 0.6$         | $0.95 \pm 0.04$                     | $12.47 \pm 0.02$       | $0.20 \pm 0.01$                     | 1.06           |
| IV   | 166             | 67  | $20.7 \pm 0.6$         | $0.81 \pm 0.04$                     | $10.44 \pm 0.03$       | $0.18 \pm 0.01$                     | 1.55           |
| V    | 129             | 60  | $14.7 \pm 0.5$         | $1.01 \pm 0.04$                     | $11.54 \pm 0.05$       | $0.35 \pm 0.02$                     | 0.37           |
| VII  | 151             | 67  | $20.0 \pm 0.5$         | $0.63 \pm 0.02$                     | $3.01 \pm 0.02$        | $0.02 \pm 0.01$                     | 0.17           |
| X    | 156             | 67  | $14.6 \pm 0.5$         | $0.91 \pm 0.04$                     | $7.77 \pm 0.05$        | $0.29 \pm 0.02$                     | 0.49           |
| XI   | 135             | 56  | $11.4 \pm 0.4$         | $0.79 \pm 0.04$                     | $7.77 \pm 0.05$        | $0.24 \pm 0.02$                     | 0.22           |

showed the worst fit ( $\sigma^2 x^2 = 3.02$ ): the high temperature reached in the plastic (real temperature  $T_1 = 87.8^\circ\text{C}$ ) made the plastic start to soften, changing the thermal and mechanical properties of the rigid bar. We omitted some other cases for brevity because we wanted to have a wide temperature difference between each of the cases studied and some of them were very close: for example, between cases V and X only  $4.2^\circ\text{C}$ , between cases VII and X only  $1^\circ\text{C}$ , and cases VIII and IX are very similar. The cooling data are better fitted by the double exponential than the corresponding steady state data, as can be seen in the last column in both tables. Case II is the exception, as a consequence of the proximity (8 mm) of the uninsulated end of the bar. The second part of eqn. (11) is verified for the cooling within statistical error, that is

$$T'_{11} + T'_{21} = T_1 \quad (15)$$

The experimental results thus wholly satisfy the theoretical auxiliary condition of eqn. (11). The differences between both the temperature and the exponent parameters are not now so large as in the steady state. The first temperature coefficient  $T'_{11}$  is 1.3–2.3 times greater than the second  $T'_{21}$  except for case VII, in which it is a factor of 6.6 greater. For the exponent parameter,  $m'_1$  is 2.1–4.8 times greater than  $m'_2$  except for case VII, in which it is 31.5 times greater, meaning that the cooling for this case tends to a single exponential law.

Table 3 lists the final exponents  $m$  and  $m'$  in the steady state and in cooling, as given by eqns. (12) and (13). The corresponding  $H$  coefficients in the steady state and in cooling are then calculated using eqns. (6) and (7) respectively. The thermal and geometrical characteristics of the plastic bar needed for these calculations are presented in Table 4. The final  $m$  and  $m'$  have the same qualitative behaviour as do the  $m$  values in Tables 1 and 2, but smoother. They can be considered constant but with large fluctuations around the mean value. The same can be said for the surface conductance

TABLE 3

Values for  $m$ ,  $m'$  and  $H$  obtained from eqns. (12), (13), (6) and (7) respectively. The last column is the right hand side in eqn. (8). The last row gives the mean values

| Case  | $m$<br>eqn. (12) | $m' \times 10^{-3}$<br>eqn. (13) | $H$<br>eqn. (6) | $H$<br>eqn. (7) | $\alpha m^2 \times 10^{-3}$<br>eqn. (8) |
|-------|------------------|----------------------------------|-----------------|-----------------|---|
| II    | 38.40            | 0.42                             | 3.55            | 8.86            | 0.17                                    |
| III   | 31.26            | 0.75                             | 2.36            | 15.83           | 0.11                                    |
| IV    | 31.89            | 0.63                             | 2.45            | 13.29           | 0.12                                    |
| V     | 31.57            | 0.66                             | 2.40            | 13.93           | 0.11                                    |
| VII   | 37.11            | 0.61                             | 3.32            | 12.87           | 0.16                                    |
| X     | 32.45            | 0.62                             | 2.54            | 13.08           | 0.12                                    |
| XI    | 36.85            | 0.55                             | 3.27            | 11.61           | 0.15                                    |
| Mean: | $34.2 \pm 2.9$   | $0.61 \pm 0.09$                  | $2.84 \pm 0.48$ | $12.8 \pm 2.0$  | $0.13 \pm 0.02$                         |

TABLE 4

Thermal and geometrical characteristics of the PMMA bar

|  |  |
|--|--|
| $c = 1.42 \times 10^3 \text{ J kg}^{-1} \text{ K}^{-1}$    | $K = 0.193 \text{ J s}^{-1} \text{ m}^{-1} \text{ K}^{-1}$ |
| $\rho = 1.190 \times 10^3 \text{ kg m}^{-3}$               | $p = 157.08 \times 10^{-3} \text{ m}$                      |
| $\alpha = 0.114 \times 10^{-6} \text{ s}^{-1} \text{ m}^2$ | $w = 196.35 \times 10^{-5} \text{ m}^2$                    |

coefficients, since they are directly calculated from the  $m$  values. The significant thing is that the values of  $H$  calculated for the steady state and for cooling are different in all cases; the second is always greater than the first by a wide range of about 2.5–6.7 times. This means that eqn. (8) is not true for the plastic, as can easily be seen by directly comparing the  $m'$  and  $\alpha m^2$  columns. The last row in Table 3 gives the mean values and the standard deviations, where the uncertainties are 8.5% and 15% for  $m$  and  $m'$  respectively, and 16–17% for  $H$ . The mean value calculated for  $H$  for the cooling is about 4.5 times the corresponding value for the steady state, and the same ratio, obviously, is found for  $\alpha m^2$  with respect to  $m'$ .

#### 4. DISCUSSION AND CONCLUSIONS

The surface conductance coefficient has always been considered difficult to determine, because of the intrinsic sources of error [1,5]. It has thus been necessary to look for a satisfactory governing equation for the temperature distribution in the steady state and in cooling. The double exponential law of eqns. (9) and (10) gives the best fit to the experimental data in both temperature regimes [4]. In some cases, in the steady state (IV, IX–XI) and in the cooling (VII), the second exponential is much smaller than the first, so that the correction is very small.

The first exponential in both regimes,  $m_1$  and  $m'_1$ , governs the sharper fall-off of the temperature, and the second,  $m_2$  and  $m'_2$ , the smoother fall for longer distances and times respectively. The  $m$  and  $m'$  coefficients obtained from eqns. (12) and (13) are the resultant slopes of a log plot of temperature versus distance or time for the whole set of experimental data in the steady state and in cooling.

The mean surface conductances, obtained from the mean values of  $m$  and  $m'$  in Table 3 and using eqns. (6) and (7) respectively, are found to be different. The  $H$  obtained in cooling (12.8) is nearly five times greater than the  $H$  obtained in the steady state (2.84). This contrasts with the results found for the iron bar [2] over the same temperature range and under the same experimental conditions. Equation (8), which was verified for the iron, is no longer valid for the plastic. Instead there are two different surface conductance coefficients,  $H$  and  $H'$ , which govern the temperature behaviour in the steady state and in cooling respectively.

It may still be questioned whether different metals and plastics (or transparent media) will have the same behaviour in their surface conductance as the iron and PMMA already studied.

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